

Crossover from the parity-conserving pair contact process with diffusion to other universality classes

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The pair contact process with diffusion (PCPD) with modulo 2 conservation (PCPD2) [$2A \rightarrow 4A$, $2A \rightarrow 0$] is studied in one dimension, focused on the crossover to other well established universality classes: the directed Ising (DI) and the directed percolation (DP). First, we show that the PCPD2 shares the critical behaviors with the PCPD, both with and without directional bias. Second, the crossover from the PCPD2 to the DI is studied by including a parity-conserving single-particle process ($A \rightarrow 3A$). We find the crossover exponent $1/\phi_1 = 0.57(3)$, which is argued to be identical to that of the PCPD-to-DP crossover by adding $A \rightarrow 2A$. This suggests that the PCPD universality class has a well-defined fixed point distinct from the DP. Third, we study the crossover from a hybrid-type reaction-diffusion process belonging to the DP [$3A \rightarrow 5A$, $2A \rightarrow 0$] to the DI by adding $A \rightarrow 3A$. We find $1/\phi_2 = 0.73(4)$ for the DP-to-DI crossover. The inequality of ϕ_1 and ϕ_2 further supports the non-DP nature of the PCPD scaling. Finally, we introduce a symmetry-breaking field in the dual spin language to study the crossover from the PCPD2 to the DP. We find $1/\phi_3 = 1.23(10)$, which is associated with a new independent route from the PCPD to the DP.

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I. INTRODUCTION

The pair contact process with diffusion (PCPD) is an interacting particle system with diffusion, pair annihilation ($2A \rightarrow 0$), and creation of offspring by pairs ($2A \rightarrow 3A$) [1–3]. It is well known that the PCPD scaling is definitely distinct from the directed percolation (DP) scaling in two dimensions and higher. However, for the one-dimensional PCPD, there is still no consensus on its critical scaling with two different viewpoints such as the DP scaling after an extremely long crossover time [4–6] and the new scaling distinct from the DP [2,7–13]. Since the critical exponents of the PCPD are so similar to those of the DP in one dimension and the strong corrections to scaling are present, it would be difficult, if not impossible, to settle down the controversy via direct numerical simulations of the PCPD. On this account, the authors suggested two critical tests in the past. One is the driven PCPD (DPCPD) [9,10] and the other is the crossover [8,7]. A recent brief review on these two approaches is available in Ref. [2]. In this paper, we continue to investigate the crossover behavior of the PCPD to other universality classes, in order to understand further the difference between the PCPD and the DP.

It is well established that the critical behavior of the PCPD is not affected by introducing modulo 2 (parity) conservation of the total particle number [12,14]. That is, even if a pair branches two offspring ($2A \rightarrow 4A$) rather than one, this model is believed to share the critical behavior with the PCPD and does not belong to the directed Ising (DI) [15–17] or the parity-conserving (PC) class [18,19]. In the context of the crossover, however, the parity-conserving PCPD provides a useful platform to study rich crossover behaviors compared to the nonconserving PCPD. Because of the special role of the parity conservation, we will refer to the

parity-conserving PCPD as PCPD2 in this paper. The term PCPD will be preserved to refer to the universality class in what follows.

We introduce a single-particle branching process in the PCPD2 to study the crossover behavior from the PCPD to other universality classes. Depending on the number of offspring in the single-particle process, one can study the crossover either to the DP (by $A \rightarrow 2A$) as in Ref. [8], or to the DI (by $A \rightarrow 3A$; see Sec. IV A). Since the PCPD is clearly independent from the DI class, we expect a nontrivial crossover scaling from the PCPD to the DI. However, this crossover scaling may not be independent from the PCPD-to-DP crossover scaling, if the PCPD universality class can be described by a well-defined fixed point and the parity conservation is irrelevant to the PCPD fixed point.

As discussed in our previous work [20] and also in Sec. IV A, the crossover scaling depends only on the initial fixed point and its crossover operator, but not directly on the terminal fixed point. The initial PCPD fixed point is blind to the parity conservation [12,14]. Therefore, in the PCPD point of view, the crossover operators associated with $A \rightarrow 2A$ and $A \rightarrow 3A$ may be undistinguishable, and thus have the same scaling dimension. Hence, we expect the same crossover exponent for the PCPD-to-DP and the PCPD-to-DI crossover scalings, which is numerically confirmed in Sec. IV A. The terminal fixed point (DP or DI), of course, depends crucially on the existence of the parity conservation in the reaction dynamics.

More complex reaction-diffusion models with hybrid-type dynamics may be considered such that $mA \rightarrow (m+k)A$ and $nA \rightarrow (n-l)A$. These models are numerically shown [12] and reasonably argued (see Sec. V in Ref. [7]) to belong to the DP in one dimension when $m > n$ and $k, l > 0$, in contrast to the cases of $m = n$ (PCPD at $m = n = 2$ and TCPD at $m = n = 3$

[12,21]). The crossover from the diffusing tp12 ($3A \rightarrow 4A, 2A \rightarrow 0$) to the DP by adding a single-particle reaction ($A \rightarrow 0$) has been studied previously [7] to find the typical “crossover” behavior between the identical (DP) universality classes as expected: a linear phase boundary and no diverging time scale. In this paper, we study the crossover from the diffusing tp22 ($3A \rightarrow 5A, 2A \rightarrow 0$) with the parity conservation to the DI by adding $A \rightarrow 3A$. A nontrivial crossover exponent for this DP-to-DI crossover is numerically estimated, which is different from the PCPD-to-DI crossover exponent. This finding again supports the distinct nature of the PCPD scaling from the DP.

When $A \rightarrow 2A$ is added to tp22, we again observe the trivial crossover between the identical (DP) classes. One may question how the tp22 can distinguish the crossover operators associated with $A \rightarrow 2A$ and $A \rightarrow 3A$, in contrast to the PCPD. The answer is simply that the parity conservation at the level of single-particle reactions is relevant to the DP fixed point, but not to the PCPD fixed point. This again further supports the difference between the DP and the PCPD fixed point even in one dimension.

The parity conservation in one dimension makes it possible to map the PCPD2 to the nonequilibrium Ising spin dynamics with the Z_2 symmetry [15,17,20]. Although this mapping can be considered nominal due to the irrelevance of the parity conservation in the PCPD, one can still ask a question regarding the effect of the symmetry-breaking field in the dual spin language. In terms of the original particle (kink) language, it implies an alternating directional bias in diffusion (or branching) of particles, which makes a pair of neighboring particles tightly bound. Then we can easily expect a simple DP-type dynamics of bound pairs ($X \rightarrow 2X, X \rightarrow 0$). Thus, with the symmetry-breaking field, we may find a new independent route from the PCPD to the DP, which is characterized by a new crossover exponent. (Similar independent routes are discussed for the DI-to-DP crossover in Ref. [20]). This is also confirmed by numerical simulations.

This paper is organized as follows: Section II introduces the dynamic rules of the PCPD2 and shows that the PCPD2 shares the critical behavior with its cousin without conservation (PCPD). By introducing the relative bias between isolated particles and pairs, the PCPD2 exhibits the same critical scaling as in the DPCPD. In Sec. III, we introduce the three crossover models by adding a single-particle reaction process to the PCPD2 and to tp22, and by adding an alternating directional bias to the PCPD2, respectively. In Sec. IV, the extensive numerical results are presented for these three models. Finally, the conclusion and the summary of the results follow in Sec. V.

II. PCPD2

This section begins with the description of the PCPD2 dynamics adopted for simulations, and then reconfirms the PCPD behavior of the PCPD2. The dynamic rules for the PCPD2 are as follows:

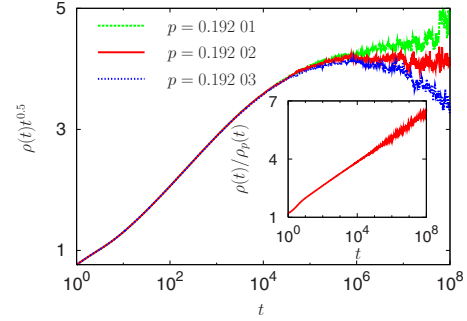
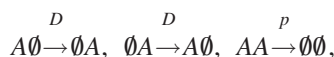
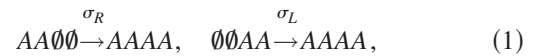


FIG. 1. (Color online) Semilogarithmic plots of $\rho(t)t^{0.5}$ vs t for the PCPD2 with the branching bias near criticality. The values of p corresponding to three curves are 0.192 01, 0.192 02, and 0.192 03, respectively, from top to bottom. Since the upper (lower) curve veers up (down) and the middle one becomes straight, the critical point is estimated as $p_c=0.192\ 02(1)$ with the number in the parentheses to be the error of the last digit. The same notation is used for numerical errors throughout the paper. Inset: Semilogarithmic plot of $\rho(t)/\rho_p(t)$ vs t at criticality ($p=0.192\ 02$). The logarithmic increase at criticality is consistent with the DPCPD behavior [9].



where A (\emptyset) stands for an occupied (vacant) site and the parameters on the top of arrows represent the transition rates. For convenience, we set $\sigma_R + \sigma_L = 1 - p$ with $0 \leq p \leq 1$. Except the biased branching model (see below), $\sigma_R = \sigma_L$ is set in most cases. Each site can carry at most one particle (hard-core exclusion).

The PCPD2 has been shown numerically to belong to the PCPD class [12,14]. This may be understood by viewing the PCPD-type models as a coupled system of two species of particles, where a particle pair corresponds to an X particle and a solitary particle to a Y particle [11,22,23]. Then, the PCPD dynamics is the combination of the DP-like dynamics of X 's and the binary annihilation diffusing dynamics of Y 's. The two species are coupled through transmutations. The only difference in the PCPD2 dynamics from the PCPD is the number of offspring in branching of X particles such as $X \rightarrow 2X$ in the PCPD and $X \rightarrow 3X$ in the PCPD2, which should be irrelevant in the DP dynamics of X 's with no parity conservation by $X \rightarrow 0$. Therefore, it is not surprising to find the PCPD2 in the PCPD class.

In this paper, we numerically study the PCPD2 with directional bias and check whether it shares the same critical behavior with the driven PCPD (DPCPD) [9,24]. This may serve as another evidence supporting that the parity conservation in the PCPD2 is irrelevant. Being relevant the relative bias between isolated particles and pairs, the biased branching as well as the biased diffusion should entail the change of the universality class [9]. Since the data with the biased branching have not been reported in the literature, we consider here the biased branching for the PCPD2 by setting $\sigma_L = 0$ and $\sigma_R = 1 - p$ in Eq. (1). The diffusion constant is set as $D = \frac{1}{2}$.

Figure 1 summarizes the simulation results using the system size 2^{23} . For each curve, 80 independent samples are

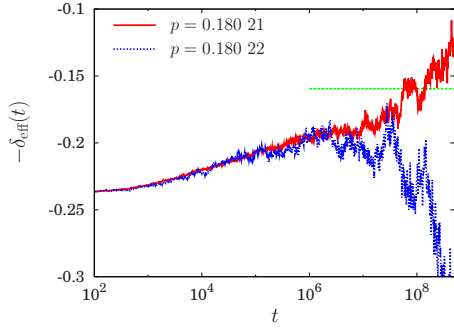


FIG. 2. (Color online) Plots of the effective exponent $-\delta_{\text{eff}}(t)$ vs t for the PCPD2 near criticality in semilogarithmic scales. For comparison, the DP value of $-\delta_{\text{DP}}$ is also shown (straight line segment). Note that the effective exponent for $p=0.180\ 21$ passes the DP value around $t=10^8$ and continues to go upwards, while that for $p=0.180\ 22$ starts to veer down already around $t=10^7$. Hence the critical point is estimated as $p=0.180\ 215(5)$.

collected. Up to the observation time, every sample has pairs, which minimally guarantees that the system does not feel the finite-size effect. During simulations, we measured the particle density $\rho(t)$, and the pair density $\rho_p(t)$. Figure 1 shows how the density decays near criticality for the PCPD2 with the biased branching ($\rho \sim t^{-\delta}$ with $\delta=1/2$ in the long-time limit at criticality) and its inset shows the logarithmic corrections to the power-law scaling. The observed critical behavior is the same as that of the DPCPD [9], which reconfirms that the parity conservation is irrelevant in the PCPD.

Now we move to the unbiased PCPD2 with $\sigma_R=\sigma_L=(1-p)/2$ in Eq. (1). In order to study the crossover behavior near the PCPD fixed point, it is necessary to measure the critical point of the PCPD2 as accurate as possible. To this end, we use the effective exponent of the density decay defined as

$$-\delta_{\text{eff}}(t) \equiv \frac{\ln[\rho(t)] - \ln[\rho(t/m)]}{\ln m}, \quad (2)$$

with $m=10$, which will drift to zero ($-\frac{1}{2}$) in the active (absorbing) phase with time and at criticality will saturate to the critical decay exponent $-\delta$ of the PCPD between 0 and $-\frac{1}{2}$. To locate the critical point, we make two assumptions. First, if $-\delta_{\text{eff}}(t)$ veers down substantially, we took this as the signal of the absorbing phase. On the other hand, if the effective exponent veers up and passes the DP value $\delta_{\text{DP}} \approx 0.1595$, we will conclude that the system is in the active phase. This must be true if δ for the PCPD is not smaller than δ_{DP} asymptotically, which has been confirmed in all of previous studies for the PCPD.

We simulated the PCPD2 at $p=0.180\ 21$ and $p=0.180\ 22$ with system size 2^{20} up to $t=5 \times 10^8$. The diffusion constant is set as $D=\frac{1}{2}$. The numbers of independent samples are 120 and 30 for $p=0.180\ 21$ and $0.180\ 22$, respectively. In these simulations, all samples survive up to the observation time in the sense that there is at least one pair. The resulting effective exponents are depicted in Fig. 2. Clearly, the PCPD2 with the annihilation probability $p=0.180\ 22$ falls into the absorbing phase. At $p=0.180\ 21$, the

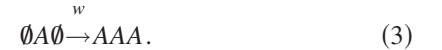
effective exponent drifts and passes well above the DP value. Hence the critical point should be located between these two values and we conclude $p_c=0.180\ 215(5)$ for the PCPD2. Note that our estimate for p_c is valid, regardless of the true asymptotic exponent value δ for the PCPD, unless δ is much smaller than δ_{DP} .

III. CROSSOVER MODELS

In this section, we introduce three models, each of which describes the PCPD-to-DI, the DP-to-DI, and the PCPD-to-DP crossover, respectively.

A. PCPD-to-DI

First, introducing the single-particle branching dynamics which preserves the parity ($A \rightarrow 3A$) to the PCPD2, the crossover from the PCPD to the DI is studied. With this single-particle dynamics, it is clear that the branching processes by pairs ($2A \rightarrow 4A$) become irrelevant (a higher-order process) and thus the system must share the critical behavior with the branching annihilating random walks with two offspring (BAW2) [25], which belongs to the DI class in one dimension. We will refer to this model as the PCPD-to-DI crossover model, or simply PCPD-DI. To be specific, together with the dynamics in Eq. (1), the single-particle branching process is included in a static fashion [26],



For a technical reason, the diffusion constant is now set as $D=(1-w)/2$ with $0 \leq w \leq 1$.

Below is the algorithm how we simulate the PCPD-DI. The Monte Carlo simulation begins with a random selection of a particle among N_t particles at time t . Let us denote the site on which the chosen particle resides by n . With probability $\frac{1}{2}$, we choose one of the direction (left or right) which will be denoted by $e \in \{1, -1\}$. If the $(n+e)$ -th site is vacant, the particle at site n hops to that site with probability $1-w$. With probability w , two offspring are newly placed at two nearest neighbors of the site n , only if the $(n-e)$ -th site is also vacant. On the other hand, if the $(n+e)$ -th site is already occupied, both particles are removed from the system with probability p or with probability $1-p$, we choose two sites in a row, that is, either $(n+2e, n+3e)$ or $(n-e, n-2e)$ with equal probability and branch two particles there once both sites are empty. If any of the conditions stated above is not satisfied, nothing happens. After the above attempt, time increases by $1/N_t$.

B. DP-to-DI

For comparison, we introduce a crossover model from the DP to the DI to check the difference of the PCPD and the DP in the crossover behavior to the DI via the same operation ($A \rightarrow 3A$). To this end, we need a DP-type model with the parity conservation. One example is the diffusing tp22 with $3A \rightarrow 5A$ and $2A \rightarrow 0$. Including $A \rightarrow 3A$, the process $3A \rightarrow 5A$ becomes irrelevant as in the PCPD2 and the model belongs to the DI class. We refer to this model as the DP-DI.

The algorithm is identical to the PCPD-DI except the triplet branching process: If the $(n+e)$ -th site is occupied, then with probability $1-p$ we look up the nearest-neighbor site of the pair in the right [site $n+(3+e)/2$]. If this site is occupied, two particles are generated at nearest neighbors of this triplet, in case both nearest-neighbor sites are vacant.

C. PCPD-to-DP via a pair-binding route

Finally, we consider an alternating diffusion bias in the one-dimensional PCPD2. All particles diffuse with bias in one direction or the other, and neighboring particles have opposite directional bias. Neighboring particles may be separated by an arbitrary stretch of empty sites. If a particle hops to the right with rate $(1+w)/2$ and to the left with $(1-w)/2$, then the hopping rates for neighboring particles are just reversed. As the particles are created or annihilated pairwise locally in Eq. (1) and the hardcore repulsion prevents the crossing of particles, each particle can preserve the bias direction and the alternating diffusion bias in the system is still intact during the dynamics. Therefore, one may identify particles with different bias as different species of particles. For example, one may name a particle with diffusion bias to the right as an A particle and a particle with diffusion bias to the left as a B particle. Then the AB ordering is preserved if one starts with a configuration with the AB ordering under the dynamics of $AB \rightarrow \emptyset\emptyset$, $AB\emptyset\emptyset \rightarrow ABAB$, $BA \rightarrow \emptyset\emptyset$, and $BA\emptyset\emptyset \rightarrow BABA$ mimicking the PCPD2 dynamics. Without bias ($w=0$), two species are indistinguishable and we recover the PCPD2.

At $w \neq 0$, this model belongs to the DP class. First, consider the extreme case ($w=1$), where A (B) always jumps to the right (left). After a transient time, all particles form AB bound pairs which cannot split once bound. Afterwards, any diffusion is impossible and the dynamics becomes identical to the contact process (CP) of $X \rightarrow 0$ and $X \rightarrow 2X$ with $X \equiv AB$. For finite w , the biased diffusion will dominate in large scales and all particles form a little bit loose but still bound AB pairs. Thus we expect the same critical behavior as in the case $w=1$. Since the CP belongs to the DP class, so does the model with $w \neq 0$. We emphasize that the fundamental difference between the PCPD and the DP lies in this pair-binding property. The crossover from the PCPD to the DP via this pair-binding route can be studied near $w=0$.

We may map the PCPD2 onto the ferromagnetic Ising spin model by interpreting a particle as a domain wall of the Ising spins [15,17,20]. Then an A and B particle represent a $\uparrow\downarrow$ and $\downarrow\uparrow$ domain wall, respectively. The alternating bias in diffusion can be interpreted as a symmetry-breaking (SB) field favoring \uparrow (\downarrow) for $w>0$ ($w<0$). So we may call the PCPD2 with finite w as the symmetry-broken PCPD2 (SB-PCPD). The crossover scaling property by the symmetry-breaking (or pair-binding) route may be different from the conventional one observed in Ref. [8], similar to various independent routes found in the DI-to-DP crossover [20]. It is interesting to note that the conventional crossover is also found by a single-particle reaction process ($A \rightarrow 2A$) breaking the parity conservation in the PCPD2.

We remark that the alternating bias (symmetry-breaking field) in the BAW2 also makes two neighboring particles

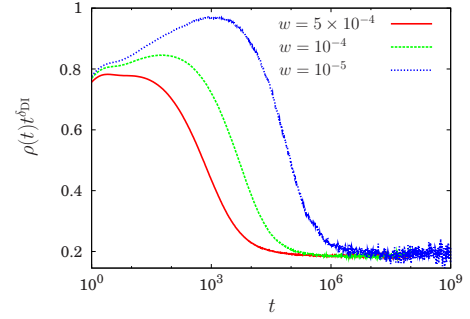


FIG. 3. (Color online) Plots of $\rho(t)t^{\delta_{DI}}$ vs t of the PCPD-DI for finite w 's at criticality in semilogarithmic scales. The smaller w is, the later the system enters the asymptotic scaling regime. Although $\delta_{DI} > \delta_{PCPD}$ is evident, the critical amplitudes do not seem to vary with w up to $w=10^{-5}$.

form a bound pair, which triggers the crossover from the DI to the DP. Therefore the pair-binding process is the common underlying mechanism for the SB routes of the DI-to-DP and the PCPD-to-DP crossover. However, the corresponding crossover scaling exponents are different, which depend on the scaling property of the initial fixed points.

D. Crossover scaling

The crossover can be described by the scaling function [8,27,28],

$$\rho(t; w, \Delta) = t^{-\delta} \mathcal{F}(\Delta^\nu t, w^\mu t), \quad (4)$$

where $\Delta = p - p_c(0)$ with $p_c(0)$ being the critical point at $w=0$ and $\mu_\parallel = \nu_\parallel / \phi$ with the crossover exponent ϕ . The exponents δ and ν_\parallel are the density decay and the relaxation-time exponent, respectively, at the initial fixed point ($w=0$).

The crossover scaling function in Eq. (4) implies that the crossover time scale diverges as $w \rightarrow 0$ such that $\tau_{\text{cross}} \sim w^{-\mu_\parallel}$ and the critical amplitude for the density decay at finite w 's should scale as $\rho t^{\delta_T} \sim w^\chi$ with $\chi = \mu_\parallel(\delta - \delta_T)$ [8], where δ_T is the density decay exponent at the terminal fixed point ($w \neq 0$). The crossover exponent ϕ can be most accurately measured from studying how the phase boundary $p_c(w)$ approaches the critical point $p_c(0)$ as $w \rightarrow 0$ such that $\Delta_c(w) \sim w^{1/\phi}$ with $\Delta_c(w) = p_c(w) - p_c(0)$.

IV. NUMERICAL RESULTS

A. PCPD-DI

This section studies the PCPD-DI, focusing on the crossover exponent ϕ . The critical point for the PCPD-DI with nonzero w can be accurately measured by using the known DI exponent value $\delta_{DI} \approx 0.285$. The DI scalings for some w 's are shown in Fig. 3 and our findings about the critical points are summarized in Table I. The crossover time scale seems to diverge as expected in the $w \rightarrow 0$ limit, which is consistent with the scaling theory prediction of Eq. (4). However, the amplitude of the critical decay ($\lim_{t \rightarrow \infty} \rho t^{\delta_{DI}}$) does not seem to vary much with w . Similar observation was reported in Ref. [8] for the crossover from the PCPD to the DP; see Fig. 2 of Ref. [8]. Of course, this does not mean $\chi=0$ or δ_{PCPD}

TABLE I. The critical points of the PCPD-DI for various w 's. The numbers in parentheses indicate the error of the last digits.

w	$p_c(w)$
0	0.180 215(5)
10^{-5}	0.182 35(5)
5×10^{-5}	0.1856(1)
10^{-4}	0.188 35(5)
2×10^{-4}	0.1922(1)
3×10^{-4}	0.1954(1)
4×10^{-4}	0.198 05(5)
5×10^{-4}	0.2005(1)

$= \delta_{DI}$. It just implies a rather big correction to scaling which results in the narrow crossover scaling regime near the PCPD fixed point [8].

Nevertheless, the crossover exponent can be quite accurately estimated from the phase boundary. Using the data for the critical points in Table I, we estimate the crossover exponents as $1/\phi=0.57(3)$, which is summarized in Fig. 4. This value of the crossover exponent is very close to that reported in Ref. [8], which studies the crossover from the PCPD to the DP via adding single-particle reaction processes. We do not think this is a mere coincidence. Rather, we believe that they are identical.

To argue the equivalence, let us first reexamine the meaning of the crossover exponent. Consider two universality classes, say I (initial) and T (terminal), and an operator triggering the crossover from the I to the T fixed points. In case of the PCPD-DI, the I fixed point resides at $[p=p_c(0), w=0]$ and the T fixed point at $[p=p_c(w^*), w=w^*]$ with $w^* > 0$, usually distant from the I fixed point. There are two relevant operators which make the I fixed point unstable. One is the crossover operator responsible for the renormalization-group (RG) flow in the direction of w and eventually into the T universality class. The other is the usual ‘‘thermal’’ operator responsible for the RG flow in the direction of the p axis.

The crossover exponent describes how the RG flow behaves locally near the I fixed point and is given by the ratio of the scaling dimensions of these two relevant operators. These scaling dimensions should be decided entirely in terms

TABLE II. The critical points of the DP-DI for various w 's.

w	p_c of the DP-DI
0	0.029 376(1)
5×10^{-6}	0.029 530(5)
10^{-5}	0.029 620(5)
2×10^{-5}	0.029 775(5)
3×10^{-5}	0.029 920(5)
5×10^{-5}	0.030 185(5)
10^{-4}	0.030 85(5)
2×10^{-4}	0.0321(1)
4×10^{-4}	0.0348(1)
5×10^{-4}	0.036 20(5)

of eigenvalues at the I fixed point [28]. Hence the T fixed point has no direct role in determining the crossover exponent. Therefore, whatever the T universality class is, the crossover exponent at the I fixed point may not change if the scaling dimensions of two relevant operators remain the same.

Now let us move on to our problem. We should emphasize that the PCPD is insensitive to the existence of the parity conservation. Therefore, the PCPD fixed point does not feel any intrinsic difference when we add one of the single-particle reaction processes of $A \rightarrow nA$ for any n . If the PCPD can be described by a well-defined fixed point, then the scaling dimension of the crossover operator associated with the $A \rightarrow nA$ process will not depend on n . Of course, the terminal (T) fixed point is sensitive to the parity conservation and thus belongs to the DP for odd n or to the DI for even n . This distinction cannot be achieved by a perturbative RG method (like the crossover exponent study), but by studying the global RG flows.

Hence, we conclude that the crossover exponent should be the same for both the PCPD-to-DP and the PCPD-to-DI crossover via a single-particle reaction route. Our numerical confirmation of this equivalence, in turn, provides another evidence that the PCPD has a well-defined fixed point distinct from the DP.

B. DP-DI

For comparison, we study the DP-DI model to study the crossover behavior from the DP to the DI mediated by the same operation ($A \rightarrow 3A$). We again confirmed that this hybrid-type multiple reaction model (the DP-DI at $w=0$) does belong to the DP class (not shown here). As before, the critical points for finite w 's are determined, exploiting the DI critical scaling. The results are summarized in Table II and in Fig. 5.

If one stops the analysis at $w=10^{-4}$, one might conclude that the phase boundary meets the vertical axis with a finite angle, and therefore the crossover exponent is 1. However, our more elaborated study with sufficiently small w 's shows that the phase boundary has a singularity with the crossover exponent $1/\phi=0.73(4)$; see Fig. 5.

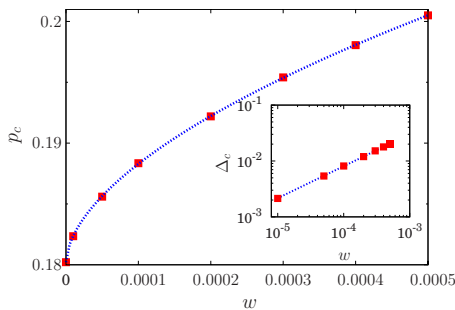


FIG. 4. (Color online) The phase boundary of the PCPD-DI near $w=0$. The curves is well fitted by the function $p_c=p_c(0)+aw^{1/\phi}$ with $1/\phi=0.57(3)$ and $a \approx 1.61$. Inset: Plot of Δ_c vs w in the logarithmic scales. The slope of the straight line is 0.57.

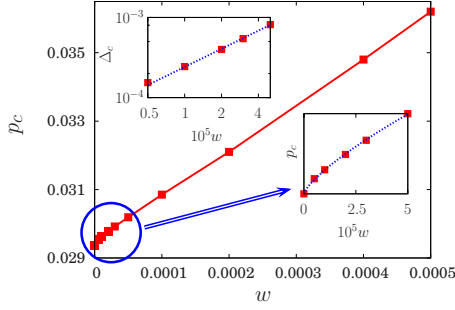


FIG. 5. (Color online) The phase boundary of the DP-DI near $w=0$. For $w \geq 10^{-4}$, the phase boundary looks almost a straight line, but the close-up of the region of $w \leq 5 \times 10^{-5}$ (panel located at the right bottom) shows a curvature which is fitted with the crossover exponent $1/\phi=0.73(4)$. The panel at the left top is the plot of Δ_c vs $10^3 w$ in the logarithmic scales. The slope of the straight line is 0.75.

Now turn back to the scaling function in Eq. (4). If we fix $\Delta=0$, the crossover scaling function takes the form

$$\rho(t)t^{\delta_{\text{DP}}} = C(w\mu_{\parallel}t), \quad (5)$$

where the numerical value of $\mu_{\parallel} = \nu_{\parallel}^{\text{DP}}/\phi = 1.26(7)$ with $\delta_{\text{DP}} \approx 0.16$ and $\nu_{\parallel}^{\text{DP}} \approx 1.73$. In Fig. 6, we depicted the scaling collapse for the particle and pair densities for $10^{-6} \leq w \leq 10^{-5}$. All curves are collapsed into a single curve in the asymptotic regime, which means that the scaling function in Eq. (5) describes the behavior near $w=0$ properly with the crossover exponent obtained before. Hence we conclude that the crossover exponent from the DP to the DI is $1/\phi = 0.73(4)$ or $\phi = 1.37(8)$ which is different from that obtained from the PCPD-DI.

C. SBPCPD

Finally, we study the SBPCPD where the alternating diffusion bias is included in the PCPD2. We expect the crossover from the PCPD to the DP via a pair-binding route, which should be independent from that via a single-particle reaction route.

The strategy to find the critical points for finite w 's is the same as before, i.e., exploiting the power-law decay of the density with the DP critical exponent δ_{DP} at criticality. The

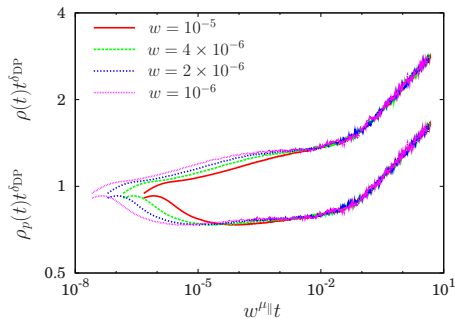


FIG. 6. (Color online) Scaling collapse for the particle (ρ) and pair densities (ρ_p) in log-log scale with $\Delta=0$; see Eq. (5). Both particle and pair densities are collapsed into a respective single curve in the asymptotic regime.

TABLE III. The critical points of the SBPCPD.

w	$p_c(w)$
0	0.180 215(5)
10^{-3}	0.180 222 5(25)
2×10^{-3}	0.180 235(5)
3×10^{-3}	0.180 250(5)
4×10^{-3}	0.180 260(5)
5×10^{-3}	0.180 275(5)
8×10^{-3}	0.180 325(5)
10^{-2}	0.180 360(10)

results are summarized in Table III. As can be seen from the table, the critical points for finite w 's are quite close to the critical point of the PCPD2. Hence, the accurate estimation of the critical points are necessary to estimate the crossover exponent with a reasonable precision. To this end, we simulated large systems ($N=2^{20}$) for long times (up to $t=2 \times 10^7$ for $w=10^{-2}$ and $t=5 \times 10^8$ for $w=10^{-3}$).

As before, we measure the crossover exponent from the singular behavior of the phase boundary. Rather than using the critical point of the PCPD2 to calculate ϕ , we fit the phase boundary using the trial function $p_c(w) = p_c(0) + a \times w^{1/\phi}$ with three fitting parameters $p_c(0)$, a , and $1/\phi$. The fitting using the whole data in Table III except $p_c(0)$ gives $p_c(0) \approx 0.180 214 7$ and $1/\phi = 1.23$. These results are summarized in Fig. 7. If we fix $p_c(0)$ between 0.182 21 and 0.182 22 and use a and $1/\phi$ as two fitting parameters, the crossover exponent varies from $1/\phi = 1.13$ to 1.33. So we estimate the crossover exponent as $1/\phi = 1.23(10)$.

For comparison, we consider the qp22 ($4A \rightarrow 6A$, $2A \rightarrow 0$), belonging to the DP class. Adding the alternating diffusion bias w (symmetry-breaking field in the dual spin language), all particles form bound pairs ($X \equiv 2A$), of which the dynamics is given by $2X \rightarrow 3X$ and $X \rightarrow 0$. Therefore, we expect the DP scaling even at nonzero w . There is no true crossover between the same (DP) universality classes and so there are no diverging time scale ($\mu_{\parallel}=0$) and also no singular behavior of the critical amplitude ($\chi=0$) as $w \rightarrow 0$. Figure 8 confirms our prediction.

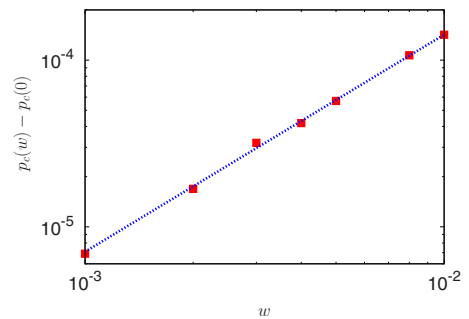


FIG. 7. (Color online) $p_c(w) - p_c(0)$ vs w plot of the SBPCPD in log-log scales. The symbols are results from simulation and the straight line is result of the fitting using the trial function $p_c(w) = p_c(0) + a \times w^{1/\phi}$ with three fitting parameters $p_c(0)$, a , and $1/\phi$. The fitting results are $p_c(0) \approx 0.180 214 7$ and $1/\phi \approx 1.23$.

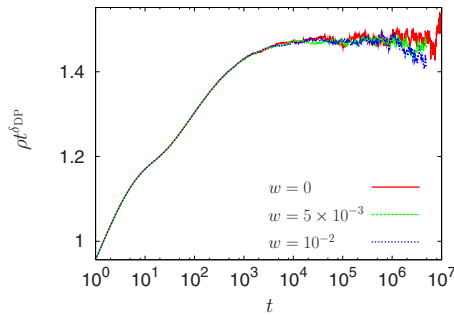


FIG. 8. (Color online) Plots of $\rho(t)t^{\phi_{DP}}$ vs t near criticality for the “crossover” model from the DP to the DP mediated by the symmetry-breaking field for $w=0$ ($p=0.031\ 405$), 5×10^{-3} ($p=0.031\ 409$), and 10^{-2} ($p=0.031\ 419$). No clear distinction among curves is observed, which is the characteristics of the “crossover” model within the same universality class.

V. SUMMARY

To summarize, we introduced and studied two types of crossover models from the pair contact process with diffusion with the parity conservation (PCPD2). We study the crossover from the PCPD by adding the parity-conserving single-particle process to the directed Ising (DI) class and by adding the alternating bias (symmetry-breaking field in the spin language) to the directed percolation (DP) class. For

comparison, we also explored the DP-to-DI crossover by studying the diffusing tp22 with the parity-conserving single-particle process.

The crossover exponents were found numerically, which read $1/\phi_1=0.57(3)$, $1/\phi_2=0.73(4)$, and $1/\phi_3=1.23(10)$, respectively, for the PCPD-to-DI, the DP-to-DI, and the PCPD-to-DP with the alternating bias. First, we argue that ϕ_1 is identical to the PCPD-to-DP via a single-particle reaction route, which is numerically confirmed. The new crossover exponent ϕ_2 found for the DP-to-DI crossover is clearly distinct from ϕ_1 , which supports the non-DP nature of the PCPD fixed point. Third, we found another independent crossover exponent ϕ_3 for the PCPD-to-DP crossover, which is ascribed to the pair-binding route different from the conventional single-particle reaction route. Nontriviality of ϕ_3 as well as ϕ_1 provides another evidence to support that the PCPD is distinct from the DP.

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